A Compositional Approach to Certifying Almost Global Asymptotic Stability of Cascade Systems

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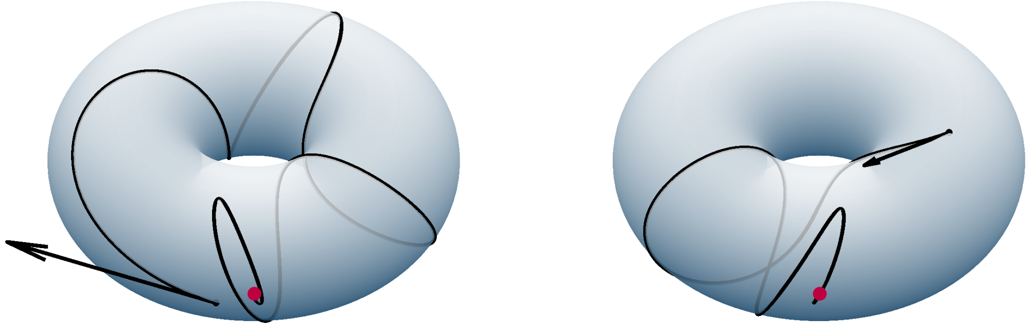

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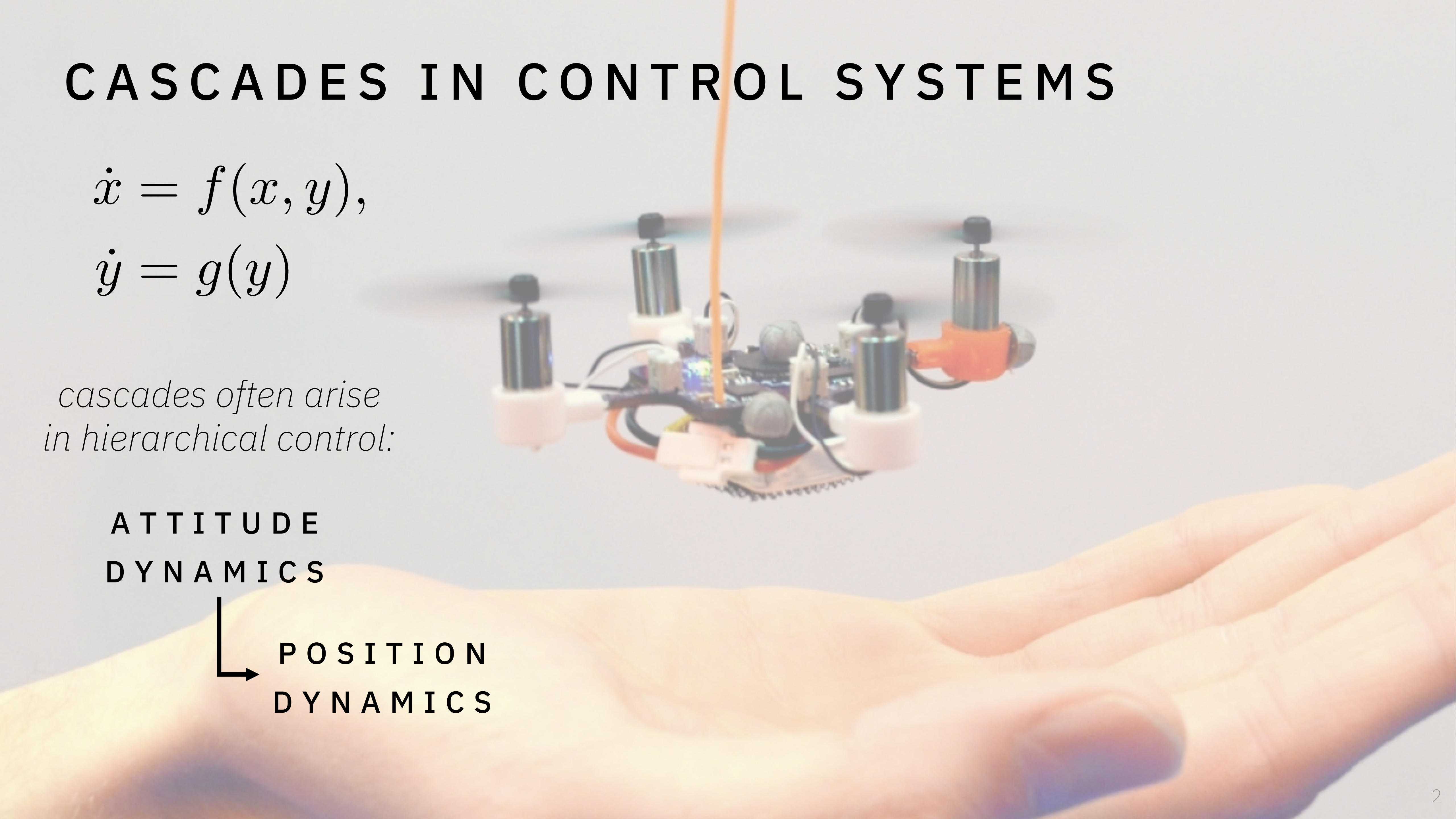
CASCADES IN CONTROL SYSTEMS



$$\dot{x} = f(x, y),$$
$$\dot{y} = g(y)$$

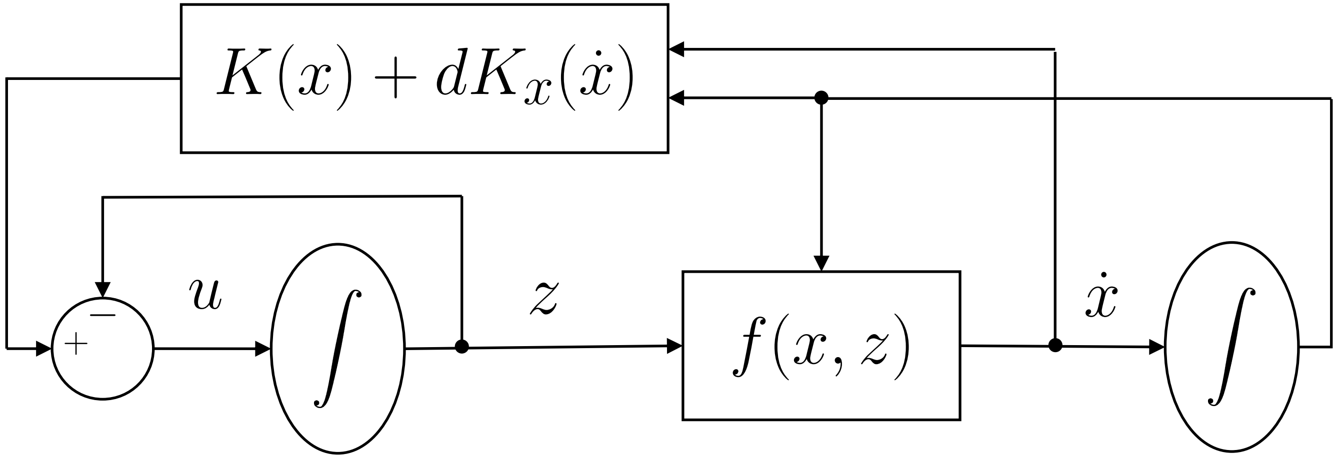
cascades often arise in hierarchical control:

ATTITUDE DYNAMICS

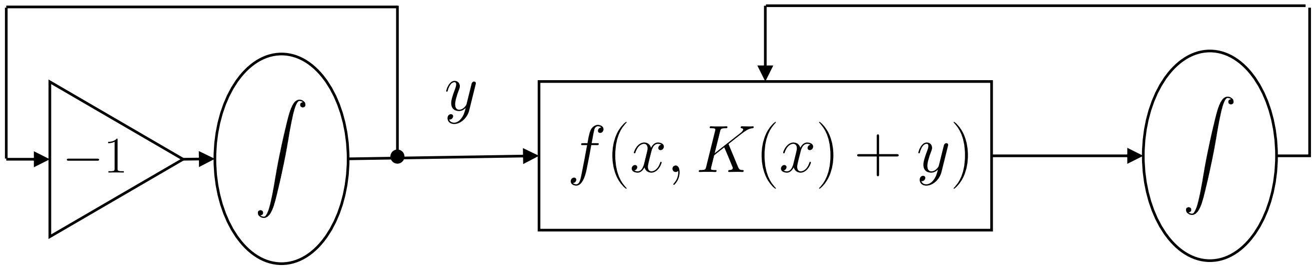
↳ **POSITION DYNAMICS**

Motivation: Hierarchical Control

Thinking of z as an input, suppose we know that $\dot{x} = f(x, K(x))$ is stable...



Letting $y = z - K(x)$ yields a cascade...



stable *to* $y = 0$

stable *when* $y = 0$

locally asymptotically stable... can we say more?

Global Asymptotic Stability of Nonlinear Cascades

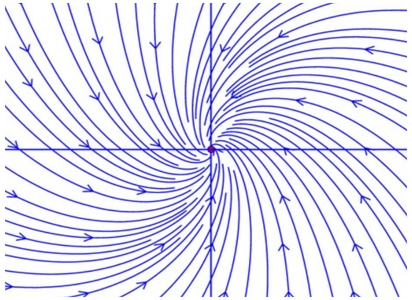
assume globally asymptotically stable *when* $y = 0$

$$\dot{x} = f(x, y),$$
$$\dot{y} = g(y)$$

assume globally asymptotically stable *to* $y = 0$

When is the combined nonlinear cascade **globally asymptotically stable**?

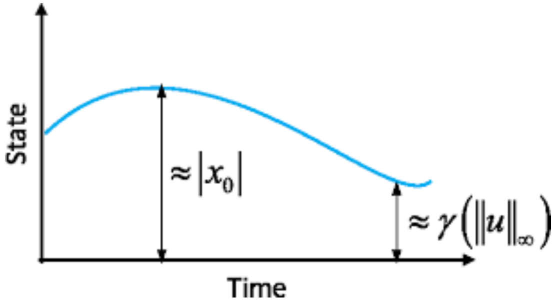
GLOBAL ASYMPTOTIC STABILITY OF SUBSYSTEMS AND BOUNDEDNESS



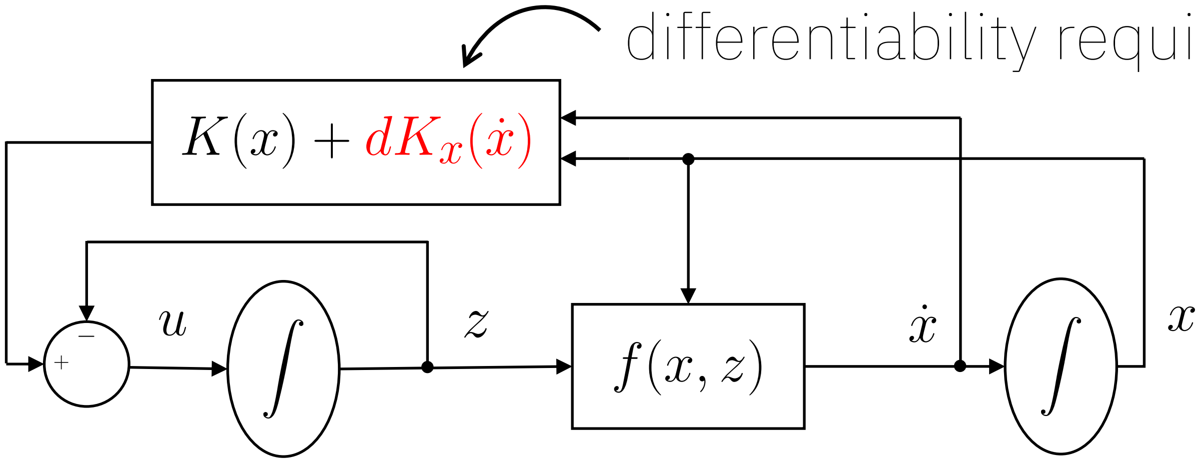
TIME SCALE SEPARATION BETWEEN SUBSYSTEMS

$$|\dot{x}| << |\dot{y}|$$

DISTURBANCE ROBUSTNESS (ISS) OF OUTER LOOP

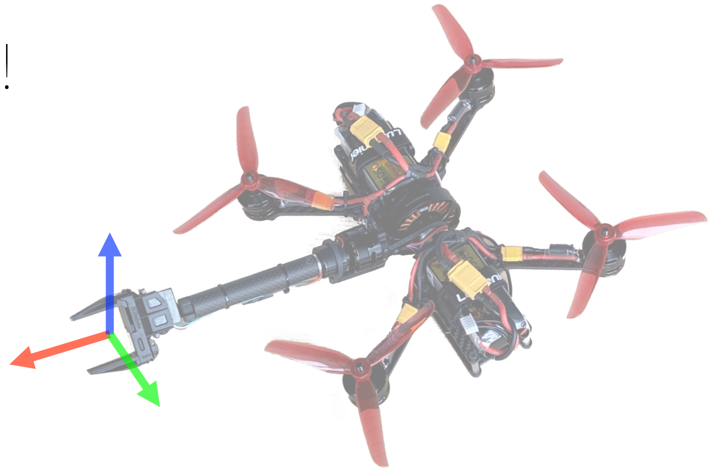


Motivation: Geometric Control of Robotic Systems



differentiability requires continuity!

HIERARCHICAL CONTROLLLER



ROBOTIC SYSTEM

For hierarchical control, we want **continuous outer loop feedback** (our intuition is that z evolves continuously, so $K(x)$ should too).

Fact. If $x \in X \not\cong \mathbb{R}^n$ and $f,\ K$ are continuous, then the stability of $\dot{x} = f(x, K(x))$ is **no better than almost global**.

Robotic systems evolve on **non-Euclidean manifolds** (e.g. $\mathbb{S}^1,\ SO(3),\ SE(3)$).

question: if the subsystems of a cascade are **almost globally asymptotically stable**, when can we say the same about the combined system?

in other words: how can we certify almost global asymptotic stability in a **compositional** manner, in order to design **verifiable hierarchical controllers**?

Simple Example System

$$x = (\theta, \dot{\theta}) \in T\mathbb{S}^1 \qquad \ddot{\theta} = -(\sin\theta + \dot{\theta})\cos 2\phi,$$

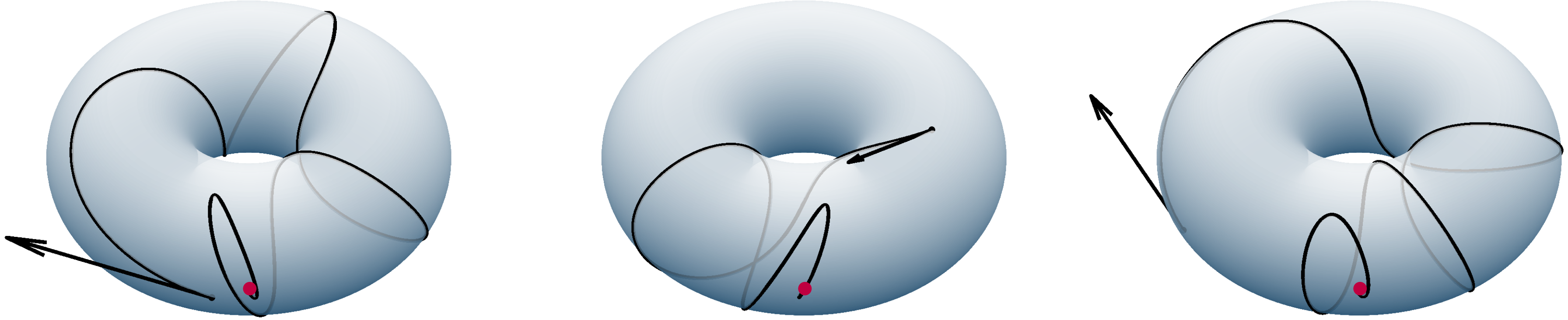
$$y = (\phi, \dot{\phi}) \in T\mathbb{S}^1 \qquad \ddot{\phi} = -(\sin\phi + \dot{\phi}) \longleftarrow$$

damped pendulum

$\ddot{\theta} = -(\sin\theta + \dot{\theta})$ *when* $\phi = 0$

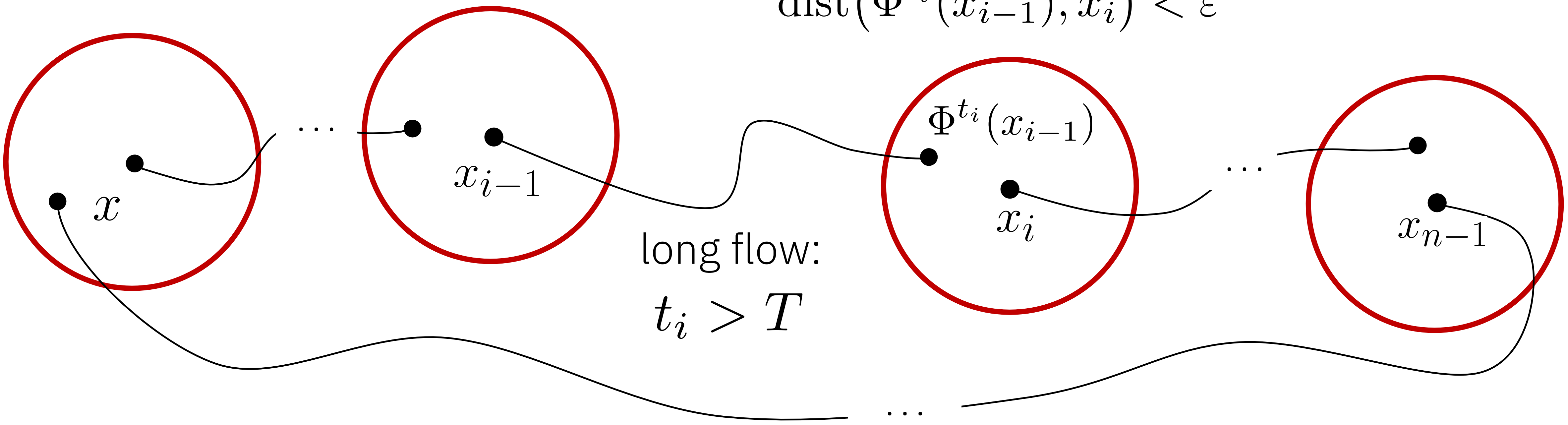
$\ddot{\theta} = +(\sin\theta + \dot{\theta})$ *when* $\phi = \frac{\pi}{2}$

subsystems are almost globally asymptotically stable.... is the full system?



- NO global asymptotic stability!
- NO time scale separation!
- NO disturbance robustness (almost ISS)!

Background: the Chain Recurrent Set of a Dynamical System



closed (ε, T)-chain:

short jumps:

$$\operatorname{dist}\left(\Phi^{t_i}(x_{i-1}), x_i\right) < \varepsilon$$

long flow:

$$t_i > T$$

x is **chain recurrent** if there exists a closed (ε, T)-chain at x for all $\varepsilon, T > 0$.

E.G. EQUILIBRIA, PERIODIC ORBITS, NON-WANDERING POINTS

Gradient-Like Dynamical Systems

A system is called **gradient-like** if all its **chain recurrent points** are **equilibria**.

Under mild assumptions, all the following are gradient-like systems:

1. **GRADIENT SYSTEMS**

$$\dot{q} = -\mathrm{grad}_{\kappa} V(q)$$

Riemannian metric *cost function*

2. **DISSIPATIVE MECHANICAL SYSTEMS**

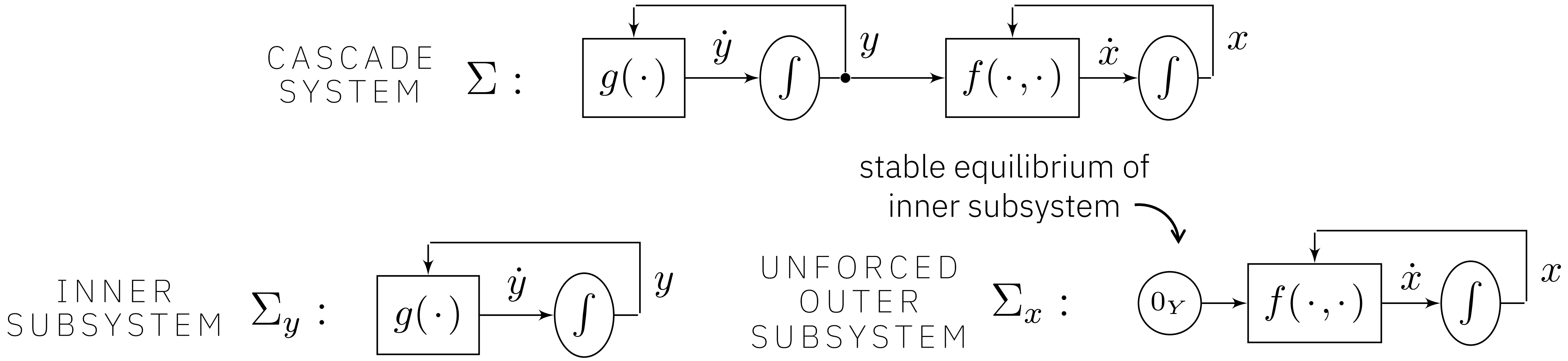
kinetic energy metric *strict Rayleigh dissipation (damping)*

$$\overset{\kappa}{\nabla}_{\dot{q}}\dot{q} = -\mathrm{grad}_{\kappa} V(q) - \kappa^{\sharp} \circ \nu^{\flat}(\dot{q})$$

potential energy

3. **GLOBALLY ASYMPTOTICALLY STABLE SYSTEMS**
4. **SYSTEMS w/ A DECREASING LYAPUNOV FUNCTION**

Main Result: Almost Global Asymptotic Stability of Cascades



CASCADE SYSTEM Σ :

stable equilibrium of inner subsystem

INNER SUBSYSTEM Σ_y :

UNFORCED OUTER SUBSYSTEM Σ_x :

Theorem *(Welde, Kvalheim, and Kumar).* Suppose that Σ_x and Σ_y are almost globally asymptotically stable, and 0_Y and all chain recurrent points of Σ_x are hyperbolic equilibria. Then, Σ is almost globally asymptotically stable and locally exponentially stable as long as all forward trajectories are bounded.

(Some of these assumptions can be relaxed; here we state a simpler result for clarity.)

Sketch of Proof for Main Result

Theorem *(Welde, Kvalheim, and Kumar).* Suppose that Σ_x and Σ_y are almost globally asymptotically stable, and 0_Y and all chain recurrent points of Σ_x are hyperbolic equilibria. Then, Σ is almost globally asymptotically stable and locally exponentially stable as long as all forward trajectories are bounded.

(Some of these assumptions can be relaxed; here we state a simpler result for clarity.)

Sketch of the Proof:

- *For each converging initial condition* $y(0)$, $\dot{x} = f(x, y(t))$ *generates an asymptotically autonomous semiflow with limit semiflow* $\dot{x} = f(x, 0_Y)$
- *Bounded trajectories of asymptotically autonomous semiflows converge to the chain recurrent set of the limit semiflow (Mischaikow, Smith and Thieme)*
- *Thus, each* $(x(t), y(t))$ *converges to some hyperbolic equilibrium* $(x^\star, 0_Y)$
- *By the stable manifold theorem, almost no solutions converge to unstable* $(x^\star, 0_Y)$

Generalization to Upper Triangular Systems

Corollary *(Welde, Kvalheim, and Kumar).* Consider an upper triangular system

$$x\left\{\dot{x}_1 = f_1(x_1, x_2, \ldots, x_n),\right.$$

$$y\left\{\begin{array}{r} \dot{x}_2 = f_2(x_2, \ldots, x_n), \\ \ddots \\ \dot{x}_n = f_n(x_n), \end{array}\right\} \textit{n-1 systems}$$

Proof: by induction!

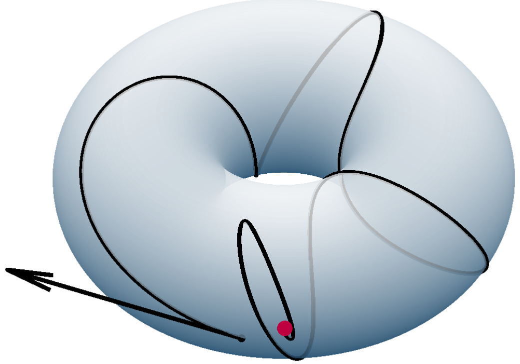
where for all $i = 1, 2, \ldots, n,$ the unforced system

$$\dot{x}_i = f_i(x_i, 0_{i+1}, 0_{i+2}, \ldots, 0_n)$$

is almost globally asymptotically stable with respect to $0_i \in X_i$ and all chain recurrent points are hyperbolic equilibria. Then, the full system is almost globally asymptotically stable and locally exponentially stable with respect to $(0_1, 0_2, \ldots, 0_n) \in X_1 \times X_2 \times \cdots \times X_n$ if all its forward trajectories are bounded.

Revisiting to the Simple Example System

$$\ddot{\theta} = -(\sin\theta + \dot{\theta})\cos 2\phi,$$
$$\ddot{\phi} = -(\sin\phi + \dot{\phi}) \longleftarrow \text{damped pendulum}$$



In fact, the system $\ddot{\phi} = -(\sin\phi + \dot{\phi})$ is dissipative mechanical for the kinetic energy and damping $\kappa = \nu = d\phi \otimes d\phi$ and potential $V : \mathbb{S}^1 \to \mathbb{R},\ \phi \mapsto 1 - \cos\phi$, so it is gradient-like i.e. all chain recurrent points are equilibria (and hyperbolic).

Theorem *(Koditschek).* A dissipative mechanical system with a strict Rayleigh dissipation and a polar Morse potential is almost globally asymptotically stable and locally exponentially stable.

Thus, our main result implies that boundedness of this system's forward trajectories will suffice for almost global asymptotic stability!

Boundedness of Cascades on Riemannian Manifolds

Theore… *…im, and Kumar).*

Suppos…

I. C…

I…

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III.

som…

are … 0_Y.

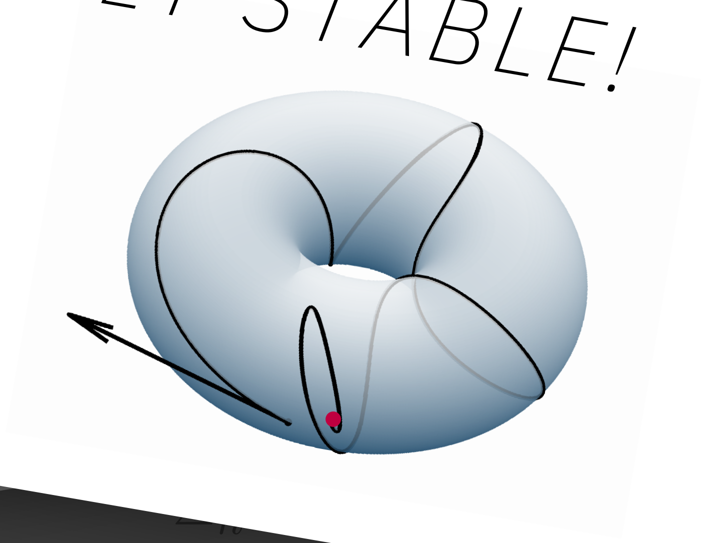
Then a… trajectory with y starting in the basin of attraction of 0_Y is bounded in forward time.

INNER S…

…BOUNDED!

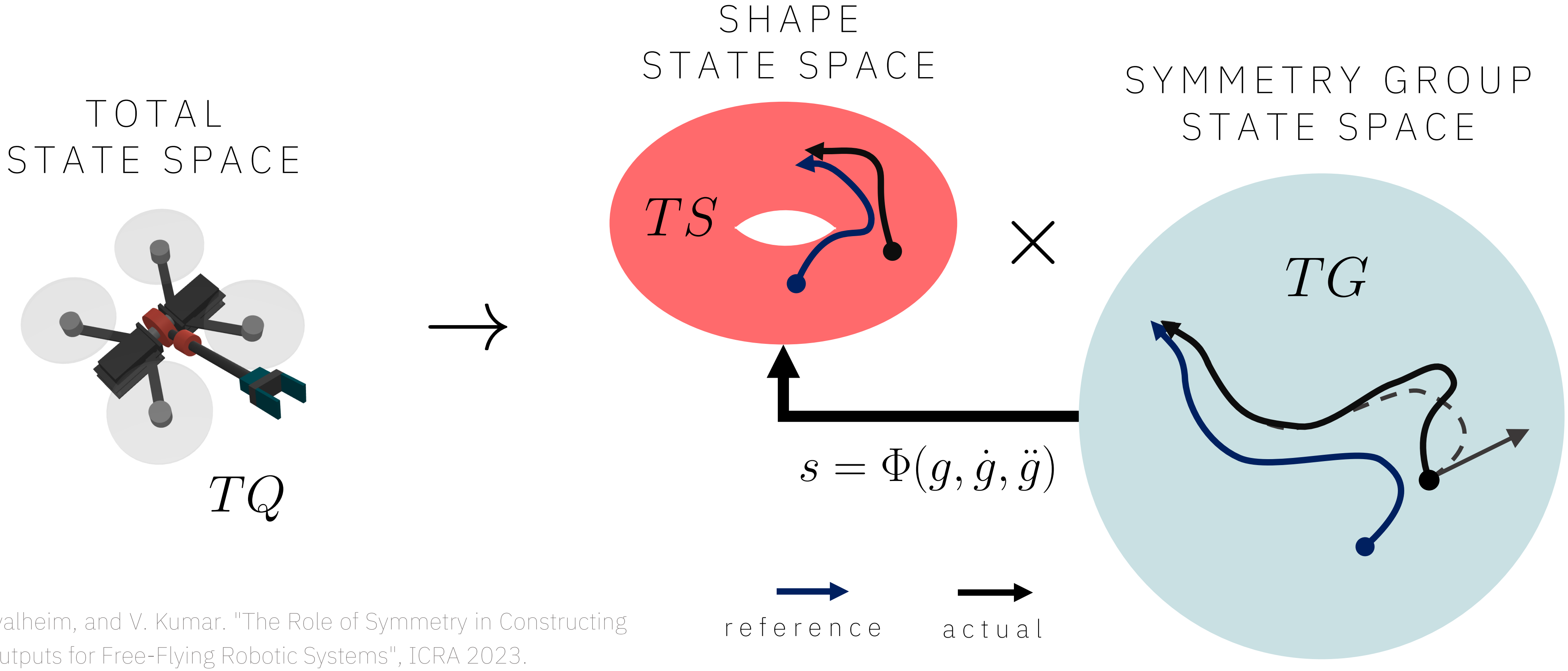
CASCADE IS ALMOST GLOBALLY ASYMPTOTICALLY STABLE!

$$\ddot{\theta} = -(\sin\theta + \dot{\theta})\cos 2\phi,$$
$$\ddot{\phi} = -(\sin\phi + \dot{\phi}),$$



Sketch of Ongoing and Future Work

question: can we use these **compositional** stability certificates to synthesize tracking controllers for a class of **underactuated robotic systems**?

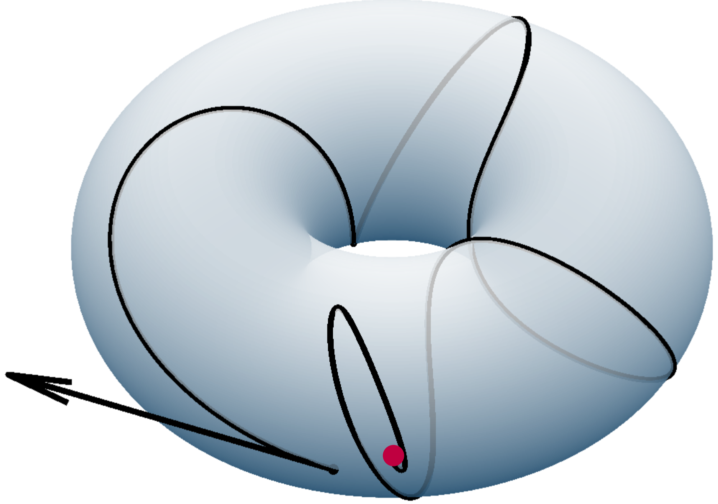


J. Welde, M.D. Kvalheim, and V. Kumar. "The Role of Symmetry in Constructing Geometric Flat Outputs for Free-Flying Robotic Systems", ICRA 2023.

In Summary

1. We give compositional sufficient conditions for almost global asymptotic stability of cascade and upper triangular systems of arbitrary size.
2. Our results constitute an almost global extension of classic global results
    a. Classic Result: GAS + GAS + Bounded => GAS
    b. Our Result: aGAS + aGAS + Bounded + “Hyperbolic Gradient-Like” => aGAS
    c. Note that for GAS systems, the only chain recurrent point is the stable equilibrium!
    d. Boundedness criteria is the Riemannian analog of Euclidean “linear growth” criteria
3. Are there more general ways to show boundedness? Further work is needed.
4. We are pursuing applications in the control of underactuated robotic systems
5. Can we extend the approach to time-varying systems?

THANKS FOR LISTENING! QUESTIONS?



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