Geometric Methods of Systematic Controller Synthesis for Underactuated Robotic Systems

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I. Motivation and Objectives

Robots must move through the world safely, capably, and predictably if we are to rely upon them to perform life-critical roles and to deliver the practical impact that our discipline has promised society. However, **the synthesis of reliable, high-performance, transferable controllers for underactuated robotic systems remains challenging**, due to the complexity and nonlinearity of the dynamics, the geometry and topology of the state space, and the diversity of robot morphologies suited for varied real-world tasks. In view of these challenges, a broadly successful solution to this problem must be:

1. **Systematic:** As the complexity and dimensionality of robotic systems grow to rival the richness of Nature, our approach to control synthesis must scale gracefully as well, renouncing handcrafted guess-and-check approaches in favor of certifiable, generalized, and trustworthy solutions.
2. **Computationally Efficient:** Agile systems like aerial and space robots are outfitted with lightweight or radiation-hardened processors. Thus, controllers must adhere to stringent computational budgets to achieve real-time operation while leaving cycles available for other autonomy tasks.
3. **Geometrically Compatible:** Robotic systems evolve on non-Euclidean manifolds subject to nonlinear, underactuated dynamics. Control algorithms must be intrinsically suited to these inherent characteristics in order to exploit the system's full hypothetical performance envelope.

While nonlinear model predictive control [4, 21] and reinforcement learning [18, 25] offer flexible languages for posing control problems, their poor **computational efficiency** often demands powerful onboard or offboard CPUs or GPUs [11, 29] or coarse approximations, *e.g.*, modeling a quadruped as a single rigid body [5] or assuming direct velocity control [12]. They have also struggled to provide **systematic** guarantees on reliability, safety, or transferability across morphologies.

Conversely, extensive research has sought to directly propose control policies with *provable* convergence, which tend to be *orders of magnitude* more **computationally efficient** than their alternatives [31]. However, such methods are often **incompatible** with the system's intrinsic geometry. Many approaches leveraging differential flatness [7, 24, 26, 17], feedback linearization [15], or constructive algorithms for nonlinear control [27] disregard the system's global geometric structure and restrict operation to a local coordinate patch of the configuration manifold. Even methods that may respect the system's geometry are not **systematic**, since they require trial and error to manually guess the requisite "flat output". It is thus common to consider either a single robot morphology [14, 17, 30, 32, 33, 34] or a narrow class under restrictive assumptions [36, 42]. Even when a cascade structure inspires the control design, it may not be fully exploited in the certificate of stability [14], prohibiting generalization to other systems with the same essential features. Such a manual approach cannot scale to complex multibody robots, whose equations of motion are rarely computed symbolically [6].

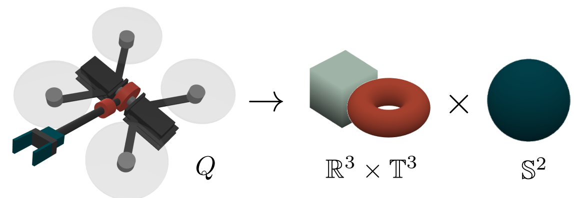

Fig. 1: We propose a systematic approach to verifiable controller synthesis for a broad class of underactuated robots, by identifying and exploiting hierarchy and constructing certificates compositionally. For this aerial manipulator, we factor the dynamics almost globally between the tangent bundles of $\mathbb{R}^3 \times \mathbb{T}^3$ (*i.e.*, the center of mass, joint angles, and yaw) and $\mathbb{S}^2$ (*i.e.*, the vehicle tilt).

To overcome these limitations, I propose **a systematic and intrinsic approach to the synthesis of efficient, verifiable controllers for underactuated robotic systems**, exploiting:

1. **Persistent Structure:** We study abstract system-level properties (*e.g.*, symmetry, differential flatness, and cascades) that *persist* across a diverse range of robot morphologies, and we develop **efficient** numerical algorithms to **systematically** identify these structures in a given robotic system.
2. **Compositionality:** We exploit such structures to *exactly* decompose the system into simpler components. With this decomposition, we **systematically** design hierarchical controllers (where the complex system-level controller is built out of simple, **efficient** subsystem controllers), all the while still *guaranteeing* the overall controller's behavior.
3. **Geometric Methods:** We develop our algorithms **natively** on the non-Euclidean manifolds where robotic systems evolve, surpassing inherently local methods by *innately* ensuring **global validity** throughout the entire state space.

Ultimately, such an approach to control synthesis will facilitate **the rapid deployment of novel robot morphologies capable of performing diverse tasks**, without the tedious retraining, manual reformulation, or laborious testing of ad-hoc methods.

II. Past Work

A. Planning with Dynamics for Underactuated Manipulators

Much of my past work has investigated which physical properties of a robotic system enable us to obtain a **decomposition** of its dynamics that facilitates control. In [39], we show that any aerial manipulator consisting of a quadrotor equipped with a manipulator arm (*e.g.*, Fig. 1) is differentially flat, and this **structure persists** even after relaxing the restrictive assumptions of [36, 34, 42] to allow for arbitrary arm geometry and degrees of freedom. Essentially, we factor the dynamics almost globally into two subsystems, where the first subsystem (the system center of mass, yaw, and joint angles) can evolve arbitrarily, while the evolution of the second subsystem (the tilt of the thrust vector) is determined uniquely by that of the first. We use this decomposition to develop an **efficient** algorithm (solving two orders of magnitude faster than, *e.g.*, [8]) to plan a dynamically feasible trajectory that exactly achieves the desired end effector motion without violating the vehicle's underactuation constraints, and to discover and characterize two especially dexterous classes of aerial manipulators.

B. Systematic Identification of Geometric Flat Outputs

Next, in [40] we generalize the decomposition used in our prior work by formally proposing the notion of a "geometric flat output", *i.e.*, a global, equivariant, Lie group-valued variant of traditional local, $\mathbb{R}^n$-valued flat outputs. Exploiting the Noetherian symmetry and Riemannian structure inherent to the dynamics of any free-flying robotic system, we derive a sufficient condition for the immediate construction of a geometric flat output. In followup work [37], we develop a finite element method (building on "optimal coordinates" for locomotion planning [10]) to apply our sufficient condition in a numerical setting. This enables the **systematic** identification of **globally** valid, symmetry-preserving numerical flat outputs, and the results achieve arbitrarily small deviation from our closed-form solutions on benchmark systems. In contrast, prior work on numerical flat output identification was purely local around a sampling of system trajectories and did not preserve symmetry [28]. Overall, our **geometric** perspective eliminates guesswork, extends flatness-based planning to a global setting, and unifies numerous results in the literature [36, 39, 17, 24, 34, 35]. By clarifying the role of symmetry in differential flatness, we also provide insight into decades-old open questions on *why* flat outputs of mechanical systems are often intuitive physical quantities (*i.e.*, a "set of points and angles" [19, 20]), such as the position of the center of mass.

C. Almost Global Asymptotic Stability of Cascades

The closed-loop dynamics of a hierarchical controller (where the state of an inner loop is thought of as the control input of an outer loop) can often be expressed in the cascade form $\dot{x} = f(x, y),\ \dot{y} = g(y)$. The stability of cascades with *globally* asymptotically stable subsystems has been studied at length [27], but topological obstructions render these methods **incompatible** with any robotic system lacking a fixed, stationary base. In particular, the outer loop feedback policy should be *continuous* in order to ensure a smooth reference for the inner loop, but an equilibrium of a continuous vector field on a non-Euclidean manifold can be no better than *almost* globally asymptotically stable ("aGAS") [2]. This begs the question: if the subsystems of a cascade are aGAS, when can we say the same about the combined system? In our recent work [41], we derive a **compositional** certificate for the aGAS of a cascade with aGAS subsystems that depends only properties of the decoupled subsystems and the "interconnection term" in isolation. Our results show that for cascades of suitable dissipative mechanical systems (a pervasive control design [13, 14, 16, 30, 38]), the only requirement for aGAS of the full cascade is the boundedness of system trajectories. Unlike prior results [1], we do not assume the subsystems have strong disturbance robustness properties often absent in practice [14].

III. Ongoing and Future Work

A. Systematic Synthesis of Geometric Tracking Controllers

Hierarchical geometric controllers have been proposed for certain flat systems on a case-by-case basis [14, 30, 33]. In our ongoing work, we instead propose a **unified** geometric tracking controller for *any* robotic system meeting our conditions for geometric flat output identification [40]. To do so, we rely on a corresponding *global* decomposition of the dynamics, abstracting away the morphological details. Then, we treat the subsystems as fully-actuated mechanical systems in cascade, and plan to verify the stability of the overall controller using our compositional certificates [41]. Inspired by [9], we design the subsystem controllers by extending existing methods restricted to systems on Lie groups [16] to the more general setting of homogeneous Riemannian manifolds [38], achieving almost global asymptotic tracking. Combined with our principled approach to flat output identification [37], this will yield a **systematic** procedure for the synthesis of formally verifiable controllers for a wide range of underactuated robotic systems, even those too complex for manual analysis.

B. "Almost Underactuated" or "Almost Flat" Systems

In Nature, hummingbirds in near-hover flight are adequately described by a "body-fixed thrust" helicopter model, but they exert some lateral thrust when performing agile maneuvers [3]. Similarly, aerial robots equipped with articulated propellers capable of high-frequency, low-amplitude thrust vectoring [22, 23] might be well-approximated on longer time-scales by underactuated models, yet require a fully-actuated model to capture their greater agility on shorter time scales. Broadly speaking, how should controllers in, *e.g.*, an "almost underactuated" regime exploit several models of varying fidelity to balance efficiency vs. expressivity tradeoffs? Additionally, even those systems that fail to *exactly* exhibit the **structures** exploited in our past and ongoing work may still be "nearby" in some rigorous sense. How might we extend our methods to a perturbed regime for systems that are, *e.g.*, "almost differentially flat", or take a cascade form "in the limit"?

C. Dynamic Physical Interaction in Hard-to-Reach Locations

Truly **dynamic** (*i.e.*, not quasistatic) physical interaction with the environment has thus far been quite limited for aerial robots working in elevated locations. We hope to ultimately endow such systems with simultaneous agility and precision, so they can perform complex, useful work quickly and reliably in locations too dangerous or hard-to-reach for human beings.

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